LOGIC AND FUZZY SYSTEMS

LOGIC

- Logic is but a small part of the human capacity to reason. Logic can be a means to compel us to infer correct answers, but it cannot by itself be responsible for our creativity or for our ability to remember.
- fuzzy logic is a method to formalize the human capacity of imprecise reasoning, or – later in this chapter – approximate reasoning.

Logic

- Part-I of this chapter introduces the reader to fuzzy logic with a review of classical logic and its operations, logical implications, and certain classical inference mechanisms such as tautologies.
- In Part-II of this chapter we introduce the use of fuzzy sets as a calculus for the interpretation of natural language.

CLASSICAL LOGIC

- In classical logic, a simple proposition P is a linguistic, or declarative, statement contained within a universe of elements, say X, that can be identified as being a collection of elements in X that are strictly true or strictly false.
- Hence, a proposition P is a collection of elements, i.e., a set, where the truth values for all elements in the set are either all true or all false.

- The veracity [və□ræsɪti:] (truth) of an element in the proposition P can be assigned a binary truth value, called *T* (P), For binary (Boolean) classical logic, *T* (P) is assigned a value of 1 (truth) or 0 (false).
- If U is the universe of all propositions,

$$T : u \in \mathrm{U} \rightarrow (0, 1)$$

All elements *u* in the universe U that are true for proposition P are called the truth set of P, denoted *T* (P). Those elements *u* in the universe U that are false for proposition P are called the falsity set of P.

CLASSICAL LOGIC

- For a universe Y and the null set $\varnothing$, we define the following truth values:
  $T(Y) = 1$ and $T(\varnothing) = 0$
- Now let P and Q be two simple propositions on the same universe of discourse that can be combined using the following five logical connectives to form logical expressions involving the two simple propositions.

Disjunction	$(\vee)$
Conjunction	$(\wedge)$
Negation	$(-)$
Implication	$(\rightarrow)$
Equivalence	$(\leftrightarrow)$

Disjunction connective

- The disjunction connective, the logical *or*, is the term used to represent what is commonly referred to as the *inclusive or*.
- The natural language term *or and the logical or* differ in that the former implies exclusion.
- the inclusive or (*logical or* as used here) implies that a compound proposition is true if either of the simple propositions is true or both are true.

Equivalence connective

- The equivalence connective arises from dual implication; that is, for some propositions P and Q,
- if $P \rightarrow Q$ and $Q \rightarrow P$, then $P \leftrightarrow Q$.

propositional calculus

- A *propositional calculus*

(sometimes called the *algebra of propositions*)

will exist for the case where proposition P measures the truth of the statement that an element, x, from the universe X is contained in set A and the truth of the statement Q that this element, x, is contained in set B, or more conventionally,

P : truth that $x \in A$

Q : truth that $x \in B$

where truth is measured in terms of the truth value, i.e.,

if $x \in A$, $T(P) = 1$; otherwise, $T(P) = 0$

if $x \in B$, $T(Q) = 1$; otherwise, $T(Q) = 0$

or, using the characteristic function to represent truth (1) and falsity (0), the following notation results:

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

mutual exclusivity

- $T(\text{P}) \cap T(\text{Q}) = \varnothing$, we have that the truth of P always implies the falsity of Q and vice versa; hence, P and Q are mutually exclusive propositions.

Example 5.1.

Let P be the proposition "The structural beam is 18WF45",

Let Q be the proposition "The structural beam is made of steel."

Let X be the universe of structural members comprised of girders, beams, and columns;

x is an element (beam), A is the set of all wide-flange (WF) beams, and B is the set of all steel beams.

Hence,

P : *x* is in A

Q : *x* is in B

Compound propositions

The five logical connectives already defined can be used to create compound propositions, where a compound proposition is defined as a logical proposition formed by logically connecting two or more simple propositions.

Just as we are interested in the truth of a simple proposition, classical logic also involves the assessment of the truth of compound propositions.

For the case of two simple propositions, the resulting compound propositions are defined next in terms of their binary truth values.

Given a proposition $P : x \in A$, $\overline{P} : x \notin A$, we have the following for the logical connectives:

Disjunction

$$P \vee Q : \; x \in A \text{ or } x \in B$$
$$\text{Hence, } T(P \vee Q) = \max(T(P), T(Q)) \tag{5.1a}$$

Conjunction

$$P \wedge Q : x \in A \text{ and } x \in B$$
$$\text{Hence, } T(P \wedge Q) = \min(T(P), T(Q)) \tag{5.1b}$$

Negation

$$\text{If } T(P) = 1, \text{ then } T(\overline{P}) = 0; \text{ if } T(P) = 0, \text{ then } T(\overline{P}) = 1. \tag{5.1c}$$

Implication

$$(P \longrightarrow Q) : x \notin A \text{ or } x \in B$$
$$\text{Hence, } T(P \longrightarrow Q) = T(\overline{P} \cup Q) \tag{5.1d}$$

Equivalence

$$(P \longleftrightarrow Q) : T(P \longleftrightarrow Q) = \begin{cases} 1, & \text{for } T(P) = T(Q) \\ 0, & \text{for } T(P) \neq T(Q) \end{cases} \tag{5.1e}$$

implication

- The logical connective *implication*, i.e., P → Q (P implies Q), presented here is also known as the classical implication.
- In this implication the proposition P is also referred to as the *hypothesis* or the *antecedent*, and the proposition Q is also referred to as the *conclusion* or the *consequent*.
- The compound proposition P → Q is true in all cases except where a true antecedent P appears with a false consequent, Q, i.e., a true hypothesis cannot imply a false conclusion.

Example

- **Example 5.2** Consider the following four propositions:

1. If 1 + 1 = 2, then 4 > 0.
2. If 1 + 1 = 3, then 4 > 0.
3. If 1 + 1 = 3, then 4 < 0.
4. If 1 + 1 = 2, then 4 < 0.

$(P \longrightarrow Q) : x \notin A \text{ or } x \in B$

Hence, $T(P \longrightarrow Q) = T(\overline{P} \cup Q)$

- The first three propositions are all true; the fourth is false. In the first two, the conclusion 4 > 0 is true regardless of the truth of the hypothesis; in the third case both propositions are false, but this does not disprove the implication; finally, in the fourth case, a true hypothesis cannot produce a false conclusion.

implication

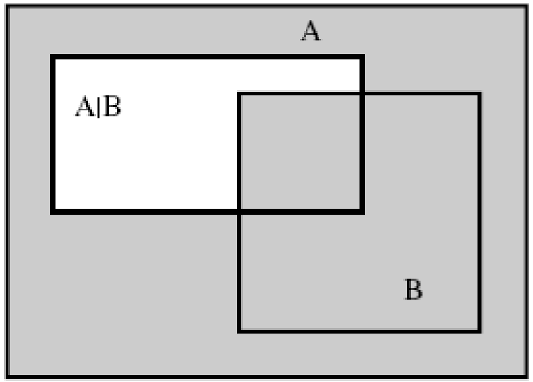
Hence, the classical form of the implication is true for all propositions of P and Q except for those propositions that are in both the truth set of P and the false set of Q, i.e.,

$$T(\mathrm{P} \longrightarrow \mathrm{Q}) = \overline{T(\mathrm{P}) \cap T(\overline{\mathrm{Q}})}$$

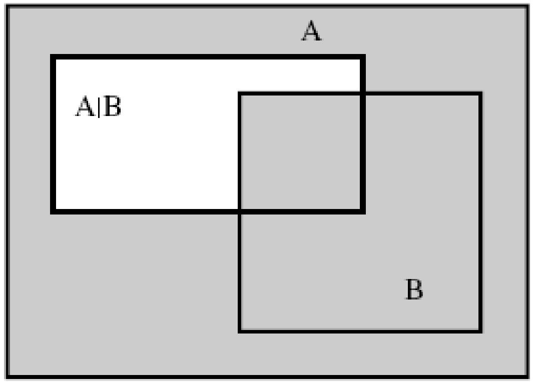
$$(\mathrm{P} \longrightarrow \mathrm{Q}) \equiv (\overline{\mathrm{A}} \cup \mathrm{B} \text{ is true}) \equiv (\text{either ``not in A'' or ``in B''})$$

so that

$$T(\mathrm{P} \longrightarrow \mathrm{Q}) = T(\overline{\mathrm{P}} \vee \mathrm{Q}) = \max(T(\overline{\mathrm{P}}), T(\mathrm{Q}))$$



For a proposition P defined on set A and a proposition Q defined on set B, the implication “P implies Q” is equivalent to taking the union of elements in the complement of set A with the elements in the set B.



This expression is linguistically equivalent to the statement, ‘‘P → Q is true’’ when either ‘‘not A’’ or ‘‘B’’ is true (logical or). Graphically, this implication and the analogous set operation are represented by the Venn diagram in Fig. 5.1. As noted in the diagram, the region represented by the difference A | B is the set region where the implication P → Q is false (the implication ‘‘fails’’). The shaded region in Fig. 5.1 represents the collection of elements in the universe where the implication is true; that is, the set

$$\overline{A \mid B} = \overline{A} \cup \overline{\overline{B}} = \overline{A \cap \overline{B}}$$

If x is in A *and* x is not in B, then

$$\mathrm{A} \longrightarrow \mathrm{B} \text{ fails} \equiv \mathrm{A} \mid \mathrm{B} \text{ (difference)}$$

TABLE 5.1
Truth table for various compound propositions

P	Q	$\overline{P}$	$P \vee Q$	$P \wedge Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T (1)	T (1)	F (0)	T (1)	T (1)	T (1)	T (1)
T (1)	F (0)	F (0)	T (1)	F (0)	F (0)	F (0)
F (0)	T (1)	T (1)	T (1)	F (0)	T (1)	F (0)
F (0)	F (0)	T (1)	F (0)	F (0)	T (1)	T (1)

$$T(P \vee Q) = \max(T(P), T(Q))$$

$$T(P \wedge Q) = \min(T(P), T(Q))$$

$$\text{If } T(P) = 1, \text{ then } T(\overline{P}) = 0$$

$$T(P \longrightarrow Q) = T(\overline{P} \vee Q) = \max(T(\overline{P}), T(Q))$$

$$T(P \longleftrightarrow Q) = \begin{cases} 1, & \text{for } T(P) = T(Q) \\ 0, & \text{for } T(P) \neq T(Q) \end{cases}$$

- Suppose the implication operation involves two different universes of discourse;
- P is a proposition described by set A, which is defined on universe X,
- Q is a proposition described by set B, which is defined on universe Y.
- Then the implication $P \rightarrow Q$ can be represented in set-theoretic terms by the relation R, where R is defined by

$$R = (A \times B) \cup (\overline{A} \times Y) \equiv \text{IF } A, \text{ THEN } B$$

$$R = (A \times B) \cup (\overline{A} \times Y) \equiv \text{IF A, THEN B}$$

IF $x \in A$ where $x \in X$ and $A \subset X$

THEN $y \in B$ where $y \in Y$ and $B \subset Y$

$$P \longrightarrow Q: \text{ IF } x \in A, \text{ THEN } y \in B, \quad \text{or} \quad P \longrightarrow Q \equiv \overline{A} \cup B$$

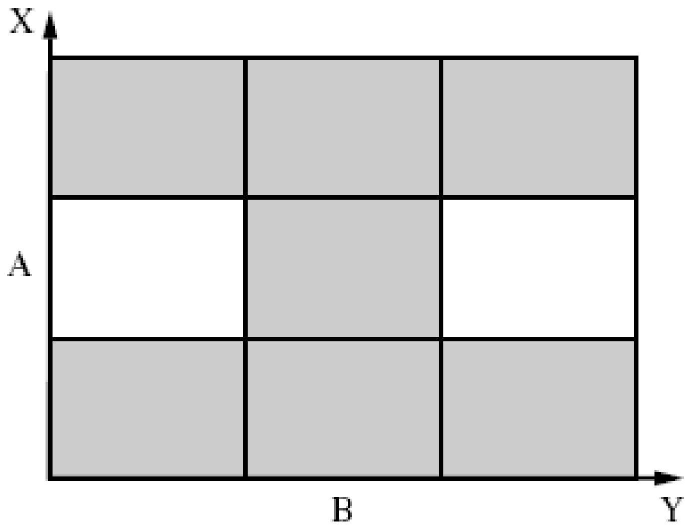


FIGURE 5.2
The Cartesian space showing the implication IF A, THEN B.

IF A, THEN B, ELSE C

Linguistically, this compound proposition could be expressed as

$$\text{IF A, THEN B,} \quad \text{and} \quad \text{IF } \overline{\text{A}}, \text{ THEN C}$$

In classical logic this rule has the form

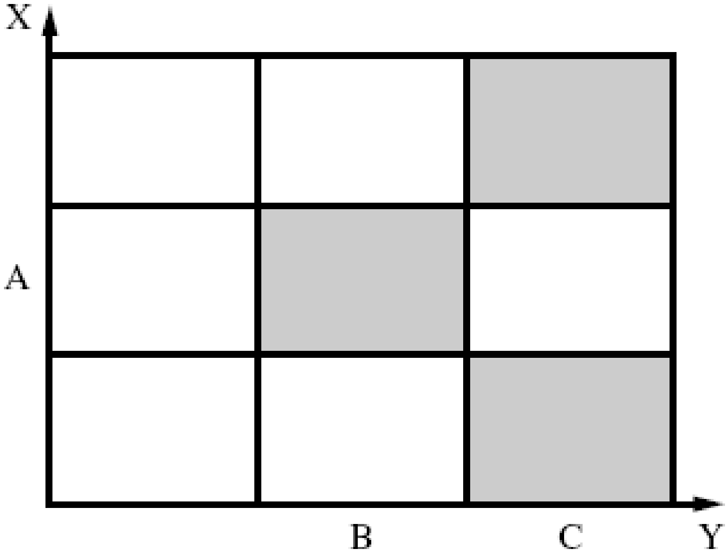
$$(P \longrightarrow Q) \wedge (\overline{P} \longrightarrow S)$$

$$P: \quad x \in A, A \subset X$$

$$Q: \quad y \in B, B \subset Y$$

$$S: \quad y \in C, C \subset Y$$

$$\text{IF A, THEN B, ELSE C} \equiv (A \times B) \cup (\overline{A} \times C) = R = \text{ relation on } X \times Y$$



Tautologies

- To consider compound propositions that are always true, irrespective of the truth values of the individual simple propositions. Classical logical compound propositions with this property are called *tautologies*.
- Tautologies are useful for deductive reasoning, for proving theorems, and for making deductive inferences.

Tautologies

- if a compound proposition can be expressed in the form of a tautology, the truth value of that compound proposition is known to be true.
- Inference schemes in expert systems often employ tautologies because tautologies are formulas that are true on logical grounds alone.
- For example, if A is the set of all prime numbers (A1 = 1, A2 = 2, A3 = 3, A4 = 5,. . .) on the real line universe, X, then the proposition "A_i is not divisible by 6" is a tautology.

modus ponens

- One tautology, known as *modus ponens* deduction, is a very common inference scheme used in forward-chaining rule-based expert systems. (It is an operation whose task is to find the truth value of a consequent in a production rule, given the truth value of the antecedent in the rule.)

- *Modus ponens* deduction concludes that, given two propositions, P and P → Q, both of which are true, then the truth of the simple proposition Q is automatically inferred.

modus tollens

- Another useful tautology is the *modus tollens* inference, which is used in backward-chaining expert systems.
- In *modus tollens* an implication between two propositions is combined with a second proposition and both are used to imply a third proposition.

Some common tautologies :

- B $\cup \overline{B} \longleftrightarrow$ X
- A $\cup$ X; $\overline{A} \cup$ X $\longleftrightarrow$ X
- (A $\wedge$ (A $\longrightarrow$ B)) $\longrightarrow$ B *(modus ponens)*
- ($\overline{B} \wedge$ (A $\longrightarrow$ B)) $\longrightarrow \overline{A}$ *(modus tollens)*

A simple proof of the truth value of the *modus ponens* deduction is provided here

Proof $(A \wedge (A \longrightarrow B)) \longrightarrow B$

$(A \wedge (\overline{A} \cup B)) \longrightarrow B$	*Implication*
$((A \wedge \overline{A}) \cup (A \wedge B)) \longrightarrow B$	*Distributivity*
$(\emptyset \cup (A \wedge B)) \longrightarrow B$	*Excluded middle axioms*
$(A \wedge B) \longrightarrow B$	*Identity*
$\overline{(A \wedge B)} \cup B$	*Implication*
$(\overline{A} \vee \overline{B}) \cup B$	*De Morgan's principles*
$\overline{A} \vee (\overline{B} \cup B)$	*Associativity*
$\overline{A} \cup X$	*Excluded middle axioms*
$X \Longrightarrow T(X) = 1$	*Identity; QED*

TABLE 5.2
Truth table (*modus ponens*)

A	B	A → B	(A ∧ (A → B))	(A ∧ (A → B)) → B	
0	0	1	0	1	
0	1	1	0	1	Tautology
1	0	0	0	1	
1	1	1	1	1	

a simple proof of the truth value of the *modus tollens* inference is listed here.

Proof

$$(\overline{B} \wedge (A \longrightarrow B)) \longrightarrow \overline{A}$$

$$(\overline{B} \wedge (\overline{A} \cup B)) \longrightarrow \overline{A}$$

$$((\overline{B} \wedge \overline{A}) \cup (\overline{B} \wedge B)) \longrightarrow \overline{A}$$

$$((\overline{B} \wedge \overline{A}) \cup \emptyset) \longrightarrow \overline{A}$$

$$(\overline{B} \wedge \overline{A}) \longrightarrow \overline{A}$$

$$\overline{(\overline{B} \wedge \overline{A})} \cup \overline{A}$$

$$(\overline{\overline{B}} \vee \overline{\overline{A}}) \cup \overline{A}$$

$$B \cup (A \cup \overline{A})$$

$$B \cup X = X \Longrightarrow T(X) = 1 \qquad \text{QED}$$

TABLE 5.3
Truth table (*modus tollens*)

A	B	$\overline{A}$	$\overline{B}$	A → B	$(\overline{B} \wedge (A \rightarrow B))$	$(\overline{B} \wedge (A \rightarrow B)) \rightarrow \overline{A}$	
0	0	1	1	1	1	1	
0	1	1	0	1	0	1	Tautology
1	0	0	1	0	0	1	
1	1	0	0	1	0	1	

Contradictions

- Compound propositions that are always false, regardless of the truth value of the individual simple propositions constituting the compound proposition, are called contradictions.
- For example, if A is the set of all prime numbers (A_1 = 1, A_2 = 2, A_3 = 3, A_4 = 5, . . .) on the real line universe, X, then the proposition "A_i is a multiple of 4" is a contradiction.
- Some simple contradictions are listed here:

$$B \cap \overline{B}$$

$$A \cap \varnothing; \quad \overline{A} \cap \varnothing$$

Equivalence

- propositions P and Q are equivalent, i.e., $P \leftrightarrow Q$, is true only when both P and Q are true or when both P and Q are false.
- For example,

P: “triangle is equilateral”

Q: “triangle is equiangular”

are equivalent because they are either both true or both false for some triangle.

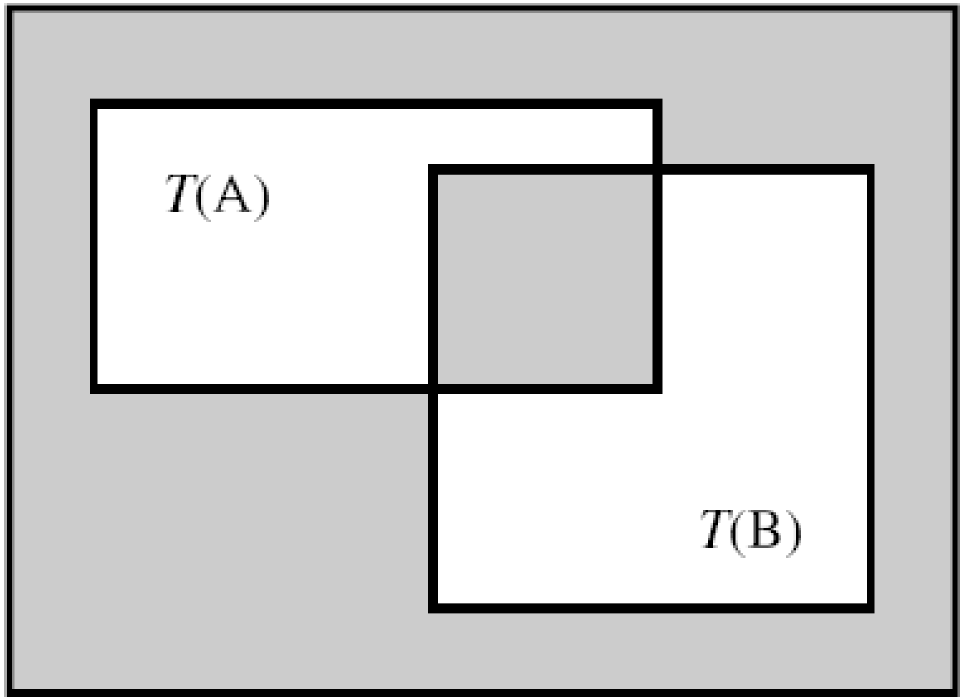


FIGURE 5.4

Venn diagram for equivalence (darkened areas), i.e., for $T(A \leftrightarrow B)$.

Example 5.3. Suppose we consider the universe of positive integers, $X = \{1 \leq n \leq 8\}$. Let $P =$ "n is an even number" and let $Q =$ "$(3 \leq n \leq 7) \wedge (n \neq 6)$." Then $T(P) = \{2, 4, 6, 8\}$ and $T(Q) = \{3, 4, 5, 7\}$. The equivalence $P \leftrightarrow Q$ has the truth set

$$T(P \longleftrightarrow Q) = (T(P) \cap T(Q)) \cup (\overline{T(P)} \cap \overline{T(Q)}) = \{4\} \cup \{1\} = \{1, 4\}$$

One can see that "1 is an even number" and "$(3 \leq 1 \leq 7) \wedge (1 \neq 6)$" are both false, and "4 is an even number" and "$(3 \leq 4 \leq 7) \wedge (4 \neq 6)$" are both true.

Example 5.4. Prove that $P \leftrightarrow Q$ if $P =$ "n is an integer power of 2 less than 7 and greater than zero" and $Q =$ "$n^2 - 6n + 8 = 0$." Since $T(P) = \{2, 4\}$ and $T(Q) = \{2, 4\}$, it follows that $P \leftrightarrow Q$ is an equivalence.

- Suppose a proposition R has the form $P \rightarrow Q$. Then the proposition $\overline{Q} \rightarrow \overline{P}$ is called the *contrapositive* of R.

- The proposition $Q \rightarrow P$ is called the *converse* of R.

- The proposition $\overline{P} \rightarrow \overline{Q}$ is called the *inverse* of R.

dual

- The *dual* of a compound proposition *that does not involve implication* is the same proposition with false (0) replacing true (1) (i.e., a set being replaced by its complement), true replacing false, conjunction ($\wedge$) replacing disjunction ($\vee$), and disjunction replacing conjunction.
- If a proposition is true, then its *dual* is also true.

Exclusive Or and Exclusive Nor

- The exclusive or:

(For example, when you are going to travel by plane or boat to some destination, the implication is that you can travel by air or sea, but not both, i.e., one or the other.)

- This situation involves the exclusive or; it does not involve the intersection.
- For two propositions, P and Q, the exclusive or, denoted here as *XOR*, is given in Table 5.4 and Fig. 5.5.

TABLE 5.4

Truth table for exclusive or, *XOR*

P	Q	P *XOR* Q
1	1	0
1	0	1
0	1	1
0	0	0

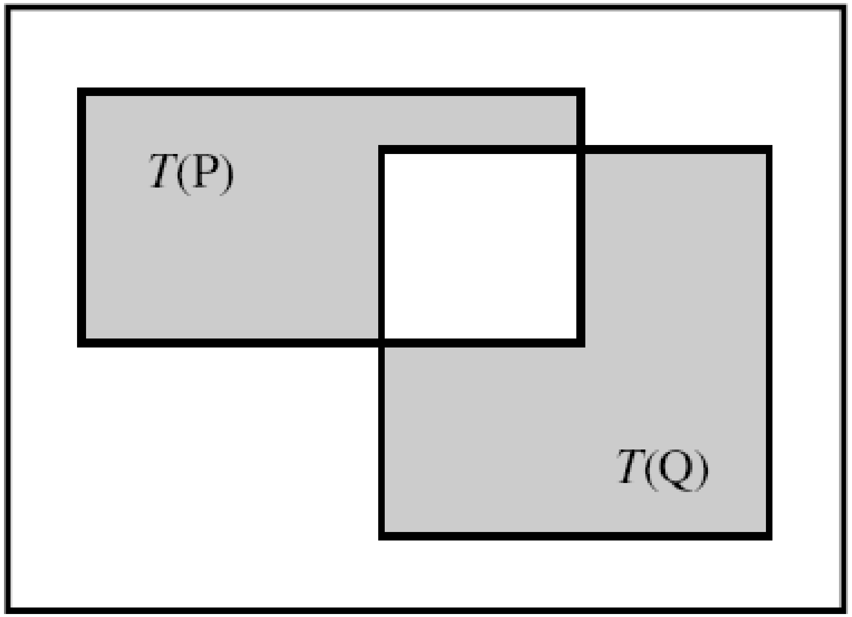


FIGURE 5.5

Exclusive or shown in gray areas.

exclusive nor

- The *exclusive nor* is the complement of the *exclusive or.*

TABLE 5.5
Truth table for exclusive nor

P	Q	$\overline{P\ XOR\ Q}$
1	1	1
1	0	0
0	1	0
0	0	1

$$\overline{P\ XOR\ Q} \longleftrightarrow (P \longleftrightarrow Q)$$

Logical Proofs

- Logic involves the use of inference in everyday life, as well as in mathematics.
- In natural language, we often inferring new facts from established facts.
- In the terminology we have been using, we want to know if the proposition

  $(P1 \wedge P2 \wedge \cdots \wedge Pn) \rightarrow Q$ is true. That is, is the statement a tautology?

The process works as follows

- First, the linguistic statement (compound proposition) is made.
- Second, the statement is decomposed into its respective single propositions.
- Third, the statement is expressed algebraically with all pertinent logical connectives in place.
- Fourth, a truth table is used to establish the veracity of the statement.

Example 5.5.

- *Hypotheses*: Engineers are mathematicians. Logical thinkers do not believe in magic. Mathematicians are logical thinkers.

  Conclusion: Engineers do not believe in magic.
- Let us decompose this information into individual propositions.

P : a person is an engineer

Q : a person is a mathematician

R : a person is a logical thinker

S : a person believes in magic

- The statements can now be expressed as algebraic propositions as

$$((P \longrightarrow Q) \wedge (R \longrightarrow \overline{S}) \wedge (Q \longrightarrow R)) \longrightarrow (P \longrightarrow \overline{S})$$

It can be shown that this compound proposition is a tautology.

Example 5.6.

Hypotheses: If an arch-dam fails, the failure is due to a poor subgrade. An arch-dam fails.
Conclusion: The arch-dam failed because of a poor subgrade.

This information can be shown to be algebraically equivalent to the expression

$$((P \longrightarrow Q) \wedge P) \longrightarrow Q$$

To prove this by contradiction, we need to show that the algebraic expression

$$((P \longrightarrow Q) \wedge P \wedge \overline{Q})$$

is a contradiction. We can do this by constructing the truth table in Table 5.6. Recall that a contradiction is indicated when the last column of a truth table is filled with zeros.

TABLE 5.6
Truth table for dam failure problem

P	Q	$\bar{P}$	$\bar{Q}$	$\bar{P} \vee Q$	$(\bar{P} \vee Q) \wedge P \wedge \bar{Q}$
0	0	1	1	1	0
0	1	1	0	1	0
1	0	0	1	0	0
1	1	0	0	1	0

Deductive Inferences

- The *modus ponens* deduction is used as a tool for making inferences in rule-based systems.
- A typical if–then rule is used to determine whether an antecedent (cause or action) infers a consequent (effect or reaction).
- Suppose we have a rule of the form IF A, THEN B, where A is a set defined on universe X and B is a set defined on universe Y.
- this rule can be translated into a relation between sets A and B; $R = (A \times B) \cup (\overline{A} \times Y)$

- Now suppose a new antecedent, say A', is known. Can we use *modus ponens* deduction to infer a new consequent, say B', resulting from the new antecedent? That is, can we deduce, in rule form, IF A', THEN B'? The answer, of course, is yes, through the use of the composition operation .
- Since "A implies B" is defined on the Cartesian space X $\times$ Y, B' can be found through the following set-theoretic formulation,

$$B' = A' \circ R = A' \circ ((A \times B) \cup (\overline{A} \times Y))$$

- *Modus ponens* deduction can also be used for the compound rule IF A, THEN B, ELSE C, where this compound rule is equivalent to the relation

$$R = (A \times B) \cup (\overline{A} \times C).$$

- For this compound rule, if we define another antecedent A', the following possibilities exist, depending on

(1) whether A' is fully contained in the original antecedent A,

(2) whether A' is contained only in the complement of A, or

(3) whether A ' and A overlap to some extent as described next:

$$\text{IF } A' \subset A, \text{ THEN } y = B$$

$$\text{IF } A' \subset \overline{A}, \text{ THEN } y = C$$

$$\text{IF } A' \cap A \neq \emptyset, \; A' \cap \overline{A} \neq \emptyset, \text{ THEN } y = B \cup C$$

The rule IF A, THEN B (proposition P is defined on set A in universe X, and proposition Q is defined on set B in universe Y), i.e., $(P \rightarrow Q) = R = (A \times B) \cup (\overline{A} \times Y)$, is then defined in function-theoretic terms as

$$\chi_R(x, y) = \max[(\chi_A(x) \wedge \chi_B(y)), ((1 - \chi_A(x)) \wedge 1)] \tag{5.9}$$

where $\chi(\,)$ is the characteristic function as defined before.

Example 5.7.

- Suppose we have two universes of discourse for a heat exchanger problem ,X = {1, 2, 3, 4} and Y = {1, 2, 3, 4, 5, 6}.Suppose X is a universe of normalized temperatures and Y is a universe of normalized pressures.
- Define crisp set A on universe X and crisp set B on universe Y : A = {2, 3} and B = {3, 4}.
- The deductive inference IF A, THEN B will yield a matrix describing the membership values of the relation R, i.e., $\chi_R(x, y)$ .That is, the matrix R represents the rule IF A, THEN B as a matrix of characteristic (crisp membership) values.

$$A = \left\{ \frac{0}{1} + \frac{1}{2} + \frac{1}{3} + \frac{0}{4} \right\}$$

$$B = \left\{ \frac{0}{1} + \frac{0}{2} + \frac{1}{3} + \frac{1}{4} + \frac{0}{5} + \frac{0}{6} \right\}$$

$$A \times B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{A} = \left\{ \frac{1}{1} + \frac{0}{2} + \frac{0}{3} + \frac{1}{4} \right\}$$

$$Y = \left\{ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right\}$$

$$\overline{A} \times Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Then the full relation R describing the implication IF A, THEN B is the maximum of the two matrices

$$A \times B \text{ and } \overline{A} \times Y$$

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

- The compound rule IF A, THEN B, ELSE C can also be defined in terms of a matrix relation as

$$R = (A \times B) \cup (\overline{A} \times C) \Rightarrow (P \rightarrow Q) \wedge (\overline{P} \rightarrow S)$$

- where the membership function is determined as

$$\chi_R(x, y) = \max[(\chi_A(x) \wedge \chi_B(y)), ((1 - \chi_A(x)) \wedge \chi_C(y))]$$

Fuzzy logic

- The restriction of classical propositional calculus to a two-valued logic has created many interesting paradoxes over the ages.
- For example, the Barber of Seville is a classic paradox (also termed Russell's barber).

In the small Spanish town of Seville, there is a rule that all and only those men who do not shave themselves are shaved by the barber. Who shaves the barber?

Barber of Seville

- Returning to the Barber of Seville, we conclude that the only way for this paradox

  (or any classic paradox for that matter) to work is if the statement is both true and false simultaneously.
- This can be shown, using set notation.

Let S be the proposition: the barber shaves himself ; $\overline{S}$ (not S) that he does not.

Then since $S \to \overline{S}$ and $\overline{S} \to S$, we have

$$T(S) = T(\overline{S}) = 1 - T(S) \qquad T(S) = \frac{1}{2}$$

- paradoxes reduce to half-truths (or half-falsities) mathematically. In classical binary (bivalued) logic, however, such conditions are not allowed, i.e., only *T(S)* = 1 or 0 is valid.
- this is a manifestation of the constraints placed on classical logic by the excluded middle axioms.
- A more subtle form of paradox can also be addressed by a multivalued logic.

- A fuzzy logic proposition, $\underline{P}$ is a statement involving some concept without clearly defined boundaries.
- Most natural language is fuzzy.
- The truth value assigned to $\underline{P}$ can be any value on the interval [0, 1].

$$T : u \in \mathrm{U} \rightarrow (0, 1)$$

$$T(\underset{\sim}{\mathrm{P}}) = \mu_{\underset{\sim}{\mathrm{A}}}(x) \quad \text{where } 0 \leq \mu_{\underset{\sim}{\mathrm{A}}} \leq 1$$

The logical connectives of negation, disjunction, conjunction, and implication are also defined for a fuzzy logic.

Negation

$$T(\overline{\underset{\sim}{\mathrm{P}}}) = 1 - T(\underset{\sim}{\mathrm{P}})$$

Disjunction

$$\underset{\sim}{\mathrm{P}} \vee \underset{\sim}{\mathrm{Q}} : x \text{ is } \underset{\sim}{\mathrm{A}} \text{ or } \underset{\sim}{\mathrm{B}} \quad T(\underset{\sim}{\mathrm{P}} \vee \underset{\sim}{\mathrm{Q}}) = \max(T(\underset{\sim}{\mathrm{P}}), T(\underset{\sim}{\mathrm{Q}}))$$

Conjunction

$$\underset{\sim}{\mathrm{P}} \wedge \underset{\sim}{\mathrm{Q}} : x \text{ is } \underset{\sim}{\mathrm{A}} \text{ and } \underset{\sim}{\mathrm{B}} \quad T(\underset{\sim}{\mathrm{P}} \wedge \underset{\sim}{\mathrm{Q}}) = \min(T(\underset{\sim}{\mathrm{P}}), T(\underset{\sim}{\mathrm{Q}}))$$

Implication [Zadeh, 1973]

$$\underset{\sim}{\mathrm{P}} \longrightarrow \underset{\sim}{\mathrm{Q}} : x \text{ is } \underset{\sim}{\mathrm{A}}, \text{ then x is } \underset{\sim}{\mathrm{B}}$$

$$T(\underset{\sim}{\mathrm{P}} \longrightarrow \underset{\sim}{\mathrm{Q}}) = T(\overline{\underset{\sim}{\mathrm{P}}} \vee \underset{\sim}{\mathrm{Q}}) = \max(T(\overline{\underset{\sim}{\mathrm{P}}}), T(\underset{\sim}{\mathrm{Q}}))$$

$$\underset{\sim}{P} \rightarrow \underset{\sim}{Q} \text{ is, IF } x \text{ is } \underset{\sim}{A}\text{, THEN } y \text{ is } \underset{\sim}{B}$$

it is equivalent to the following fuzzy relation,

$$\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times Y)$$

$$\mu_{\underset{\sim}{R}}(x, y) = \max[(\mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(y)), (1 - \mu_{\underset{\sim}{A}}(x))]$$

Example 5.9. Suppose we are evaluating a new invention to determine its commercial potential. We will use two metrics to make our decisions regarding the innovation of the idea. Our metrics are the "uniqueness" of the invention, denoted by a universe of novelty scales, $X = \{1, 2, 3, 4\}$, and the "market size" of the invention's commercial market, denoted on a universe of scaled market sizes, $Y = \{1, 2, 3, 4, 5, 6\}$. In both universes the lowest numbers are the "highest uniqueness" and the "largest market," respectively. A new invention in your group, say a compressible liquid of very useful temperature and viscosity conditions, has just received scores of "medium uniqueness," denoted by fuzzy set $\underset{\sim}{A}$, and "medium market size," denoted fuzzy set $\underset{\sim}{B}$. We wish to determine the implication of such a result, i.e., IF $\underset{\sim}{A}$, THEN $\underset{\sim}{B}$. We assign the invention the following fuzzy sets to represent its ratings:

$$\underset{\sim}{A} = \text{medium uniqueness} = \left\{\frac{0.6}{2} + \frac{1}{3} + \frac{0.2}{4}\right\}$$

$$\underset{\sim}{B} = \text{medium market size} = \left\{\frac{0.4}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.3}{5}\right\}$$

$$\underset{\sim}{C} = \text{diffuse market size} = \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.3}{6}\right\}$$

$$\underset{\sim}{A} \times \underset{\sim}{B} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \end{bmatrix} \end{array} \left[\begin{pmatrix} 0 \\ 0.6 \\ 1 \\ 0.2 \end{pmatrix} (0 \quad 0.4 \quad 1 \quad 0.8 \quad 0.3) \right] \quad \textit{take} \min$$

$$\overline{\underset{\sim}{A}} \times Y = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

$$\underset{\sim}{R} = \max(\underset{\sim}{A} \times \underset{\sim}{B}, \overline{\underset{\sim}{A}} \times Y)$$

$$\underset{\sim}{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 0.4 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

$$\text{IF } x \text{ is } \underset{\sim}{A}, \text{ THEN } y \text{ is } \underset{\sim}{B}, \text{ ELSE } y \text{ is } \underset{\sim}{C}$$

$$\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times \underset{\sim}{C})$$

$$\mu_{\underset{\sim}{R}}(x, y) = \max\left[(\mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(y)), ((1 - \mu_{\underset{\sim}{A}}(x)) \wedge \mu_{\underset{\sim}{C}}(y))\right]$$

$$\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times \underset{\sim}{C}) \qquad \overline{\underset{\sim}{A}} \times \underset{\sim}{C} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.4 & 0.4 & 0.4 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \end{bmatrix} \end{array}$$

$$\underset{\sim}{A} \times \underset{\sim}{B} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \end{bmatrix} \end{array} \qquad \underset{\sim}{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.6 & 0.6 & 0.4 & 0.3 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \end{bmatrix} \end{array}$$

APPROXIMATE REASONING

- The ultimate goal of fuzzy logic is to form the theoretical foundation for reasoning about imprecise propositions; such reasoning has been referred to as approximate reasoning.

Rule 1: IF x is $\underset{\sim}{A}$, THEN y is $\underset{\sim}{B}$, where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ represent fuzzy propositions (sets). Now suppose we introduce a new antecedent, say $\underset{\sim}{A}'$, and we consider the following rule:

Rule 2: IF x is $\underset{\sim}{A}'$, THEN y is $\underset{\sim}{B}'$.

From information derived from Rule 1, is it possible to derive the consequent in Rule 2, $\underset{\sim}{B}'$? The answer is yes, and the procedure is fuzzy composition. The consequent $\underset{\sim}{B}'$ can be found from the composition operation, $\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R}$.

The two most common forms of the composition operator are the max–min and the max–product compositions, as initially defined in Chapter 3.

Example 5.10. Continuing with the invention example, Example 5.9, suppose that the fuzzy relation just developed, i.e., $\underset{\sim}{R}$, describes the invention's commercial potential. We wish to know what market size would be associated with a uniqueness score of "almost high uniqueness." That is, with a new antecedent, $\underset{\sim}{A}'$, the following consequent, $\underset{\sim}{B}'$, can be determined using composition. Let

$$\underset{\sim}{A}' = \text{almost high uniqueness} = \left\{ \frac{0.5}{1} + \frac{1}{2} + \frac{0.3}{3} + \frac{0}{4} \right\}$$

Then, using the following max–min composition,

$$\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R} = \left\{ \frac{0.5}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.5}{6} \right\}$$

we get the fuzzy set describing the associated market size. In other words, the consequent is fairly diffuse, where there is no strong (or weak) membership value for any of the market size scores (i.e., no membership values near 0 or 1).

$$\underset{\sim}{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 0.4 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

0.5 1 0.3 0

min 1 0.4 0 0.8

0.5 0.4 0 0

max:0.5

- An interesting issue in approximate reasoning is the idea of an inverse relationship between fuzzy antecedents and fuzzy consequences arising from the composition operation.
- Suppose we use the original antecedent, A, in the fuzzy composition. Do we get the original fuzzy consequent, B, as a result of the operation? That is, does the composition operation have a unique inverse, i.e., $B = A \circ R$? The answer is an unqualified no, and one should not expect an inverse to exist for fuzzy composition.

Example 5.12. Again, continuing with the invention example, Examples 5.9 and 5.10, suppose that $\underset{\sim}{A}' = \underset{\sim}{A} =$ "medium uniqueness." Then

$$\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R} = \underset{\sim}{A} \circ \underset{\sim}{R} = \left\{ \frac{0.4}{1} + \frac{0.4}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.4}{5} + \frac{0.4}{6} \right\} \neq \underset{\sim}{B}$$

That is, the new consequent does not yield the original consequent ($\underset{\sim}{B} =$ medium market size) because the inverse is not guaranteed with fuzzy composition.

In classical binary logic this inverse does exist; that is, crisp modus ponens would give

$$B = A \circ R = A \circ R = B,$$

where the sets A and B are crisp, and the relation R is also crisp.

Example 5.13. Suppose you are a soils engineer and you wish to track the movement of soil particles under applied loading in an experimental apparatus that allows viewing of the soil motion. You are building pattern recognition software to enable a computer to monitor and detect the motions. However, there are some difficulties in "teaching" your software to view the motion. The tracked particle can be occluded by another particle. The occlusion can occur when a tracked particle is behind another particle, behind a mark on the camera's lens, or partially out of sight of the camera. We want to establish a relationship between particle occlusion, which is a poorly known phenomenon, and lens occlusion, which is quite well-known in photography. Let these membership functions,

$$\underset{\sim}{A} = \left\{ \frac{0.1}{x_1} + \frac{0.9}{x_2} + \frac{0.0}{x_3} \right\} \quad \text{and} \quad \underset{\sim}{B} = \left\{ \frac{0}{y_1} + \frac{1}{y_2} + \frac{0}{y_3} \right\}$$

describe fuzzy sets for a *tracked particle moderately occluded* behind another particle and a *lens mark associated with moderate image quality*, respectively. Fuzzy set $\underset{\sim}{A}$ is defined on a universe $X = \{x_1, x_2, x_3\}$ of tracked particle indicators, and fuzzy set $\underset{\sim}{B}$ (note in this case that $\underset{\sim}{B}$ is a crisp singleton) is defined on a universe $Y = \{y_1, y_2, y_3\}$ of lens obstruction indices. A typical rule might be: IF occlusion due to particle occlusion is moderate, THEN image quality will be similar to a moderate lens obstruction, or symbolically,

$$\text{IF } x \text{ is } \underset{\sim}{A}, \text{ THEN } y \text{ is } \underset{\sim}{B} \text{ or } (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times Y) = \underset{\sim}{R}$$

We can find the relation, $\underset{\sim}{R}$, as follows:

$$\underset{\sim}{A} \times \underset{\sim}{B} = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0 & 0.1 & 0 \\ x_2 & 0 & 0.9 & 0 \\ x_3 & 0 & 0 & 0 \end{matrix} \qquad \overline{\underset{\sim}{A}} \times \underset{\sim}{Y} = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0.9 & 0.9 & 0.9 \\ x_2 & 0.1 & 0.1 & 0.1 \\ x_3 & 1 & 1 & 1 \end{matrix}$$

$$\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times \underset{\sim}{Y}) = \begin{bmatrix} 0.9 & 0.9 & 0.9 \\ 0.1 & 0.9 & 0.1 \\ 1 & 1 & 1 \end{bmatrix}$$

This relation expresses in matrix form all the knowledge embedded in the implication. Let $\underset{\sim}{A}'$ be a fuzzy set, in which a tracked particle is behind a particle with *slightly more occlusion* than the particle expressed in the original antecedent $\underset{\sim}{A}$, given by

$$\underset{\sim}{A}' = \left\{ \frac{0.3}{x_1} + \frac{1.0}{x_2} + \frac{0.0}{x_3} \right\}$$

We can find the associated membership of the image quality using max–min composition. For example, approximate reasoning will provide

$$\text{IF } x \text{ is } \underset{\sim}{A}', \ \text{THEN } \underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R}$$

and we get

$$\underset{\sim}{B}' = [0.3 \ 1 \ 0] \circ \begin{bmatrix} 0.9 & 0.9 & 0.9 \\ 0.1 & 0.9 & 0.1 \\ 1 & 1 & 1 \end{bmatrix} = \left\{ \frac{0.3}{y_1} + \frac{0.9}{y_2} + \frac{0.3}{y_3} \right\}$$

This image quality, $\underset{\sim}{B}'$, is more fuzzy than $\underset{\sim}{B}$, as indicated by the former's membership function.

Other forms of the implication operation

There are other techniques for obtaining the fuzzy relation $\underset{\sim}{R}$ based on the IF $\underset{\sim}{A}$, THEN $\underset{\sim}{B}$, or $\underset{\sim}{R} = \underset{\sim}{A} \rightarrow \underset{\sim}{B}$. These are known as fuzzy implication operations, and they are valid for all values of $x \in X$ and $y \in Y$.

values of $x \in X$ and $y \in Y$.

$$\mu_{\underset{\sim}{R}}(x, y) = \max[\mu_{\underset{\sim}{B}}(y), 1 - \mu_{\underset{\sim}{A}}(x)]$$

$$\mu_{\underset{\sim}{R}}(x, y) = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)]$$

$$\mu_{\underset{\sim}{R}}(x, y) = \min\{1, [1 - \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(y)]\}$$

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(y)$$

$$\mu_{\underset{\sim}{R}}(x, y) = \begin{cases} 1, & \text{for } \mu_{\underset{\sim}{A}}(x) \le \mu_{\underset{\sim}{B}}(y) \\ \mu_{\underset{\sim}{B}}(y), & \text{otherwise} \end{cases}$$